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Optimal Interpolation in a General Setting— A Beginning

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INTRODUCTION

Let the subspace $\mathbf{Y}$ of $C[a, b]$ of dimension $n+1$ $(n \geqslant 0)$ be spanned by a complete extended Tchebycheff system $u_0, ..., u_n$. It is possible to find for given nodes $t_0, ..., t_n$

$$a = t_0 < \cdots < t_n = b,$$

a basis $\{y_0, ..., t_n\}$ of $\mathbf{Y}$ such that

$$y_i(t_j) = \delta_{ij} \qquad \text{(Kronecker delta)},$$

and to obtain an interpolating projection

$$P\colon C[a, b] \to \mathbf{Y}$$

defined by

$$Pf = \sum_{i=0}^{n} f(t_i)\, y_i.$$

It is seen that

$$\|P\| = \left\| \sum_{i=0}^{n} |y_i| \right\|.$$

The function

$$L = \sum_{i=0}^{n} |y_i|$$

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has the properties that

> $L(t)=1$ if t is a node, and if $1 \leqslant i \leqslant n$, L is maximized on $[t_{i-1}, t_i]$ at a unique point T_i and $t_{i-1} < T_i < t_i$, $L'(T_i)=0$.

Clearly, if $n \geqslant 1$ $\|P\| = \max\{L(T_1), ..., L(T_n)\}$, and $\|P\|$ depends upon the choice of the nodes $t_1, ..., t_{n-1}$.

It was conjectured in [8] that, under the hypotheses laid down above on $\mathbf{Y}$ and P, the norm of P is minimal if

$$L(T_1) = \cdots = L(T_n) = C_Y, \qquad \text{for some unique value } C_Y$$

(generalization of the Bernstein conjecture regarding Lagrange interpolation [1]), and furthermore if $\|P\| > C_Y$, at least one of the local maximum values of L is less than C_Y (generalization of the Erdös conjecture on Lagrange interpolation [5]). One strong argument for the plausibility of this new conjecture is, of course, the upholding of the two original conjectures in their original context as theorems. Optimal Lagrange interpolation is indeed characterized by their conditions [6, 2, 7]. In the above-mentioned article [8], it was also pointed out that two impediments existed to the immediate successful generalization of these two conjectures, although there are at this time several choices of the range space $\mathbf{Y}$ for which the above generalized conjectures are indeed valid [2, 7, 8, 9, 10], as well as examples of interpolation with spaces of functions on a complex domain for which the conditions of Bernstein and Erdös have been shown to characterize minimal norm interpolation [3, 4]. It is the purpose of this communication to lay down a solution of the first of these two difficulties, in the framework of which there may be a possibility of overcoming the second.

An Overview of the Problem

In the context described above, we let

$$\lambda_i = L(T_i) = \max_{[t_{i-1}, t_i]} L, \qquad i \in \{1, ..., n\},$$

and we let the function in $\mathbf{Y}$ which agrees with L on the interval $[t_{i-1}, t_i]$ be denoted by X_i. Finally, we denote by T_i the local maximum on the interval $[t_{i-1}, t_i]$, again for $i \in \{1, ..., n\}$. The derivative of the function

$$(t_1, ..., t_{n-1}) \mapsto (\lambda_1, ..., \lambda_n)$$

exists and is given by a matrix

$$\left(\frac{\partial \lambda_i}{\partial t_j}\right)_{i=1}^{n}{}_{j=1}^{n-1}, \tag{1}$$

in which we will let the index i represent columns and the index j represent rows in the following discussion. We denote by J_p the determinant of the square matrix derived by removing the pth column, for each $p \in \{1, ..., n\}$. To establish the generalized Bernstein and Erdös conjectures of [8] as valid characterizations of optimal interpolation into $\mathbf{Y}$, it suffices to show that

(i) $J_p \neq 0$ for all possible choices of the nodes and for

$$p \in \{1, ..., n\},$$

and

(ii) J_p alternates in sign.

The first step of proof is the establshment of a more explicit expression for the entries of the derivative matrix. We have in previous cases and also in this general context the equivalence

$$\frac{\partial \lambda_i}{\partial t_j} = -y_j(T_i)\, X_i'(t_j). \tag{2}$$

The second step, and the first difficulty in the way of generalization, mentioned in [8], is the reduction of the matrix by certain column and row cancellations, to an equivalent matrix

$$(q_i(t_j))_{i=1}^{n}{}_{j=1}^{n-1}, \tag{3}$$

reducing (i) and (ii) to a question of whether the set of functions $\{q_1, ..., q_n\} \backslash \{q_p\}$, $p \in \{1, ..., n\}$, admits a non-trivial linear combination which is zero on the points $t_1, ..., t_{n-1}$. In the next section of this communication, a general method for carrying out this crucial step of reduction of the problem to a question about a matrix of form (3) will be laid down. The final step in the argument is, of course, the last-mentioned difficulty of [8], namely that of showing that the question about a matrix of form (3) can be answered in the particular case under consideration. Some advance has been made to date on this aspect of the problem as well [9, 10], and the results presented here make it apparent that the characterization of optimal interpolation with much broader classes of functions by the Bernstein and Erdös criteria would immediately follow from progress on this front. We now present the results.

THEOREM. *Let* $\mathbf{Y}$ *be an* $n+1$ *dimensional subspace of* $C[a, b]$ *which is spanned by an extended complete Tchebycheff system, and let a system of nodes* $t_0, ..., t_n$ *be chosen as described above, and an interpolating projection built on these nodes. Then to the matrix denoted by* (1) *corresponds a matrix of evaluation of the form* (3).

COROLLARY. *If the space* $\mathbf{Y}$ *is a space of polynomials, the corresponding matrix of form* (3) *can be written such that the functions* $q_1, ..., q_n$ *are themselves polynomials.*

Proof of the theorem. Given a set of basis functions $\{u_0, ..., u_n\}$ for $\mathbf{Y}$ and a set of nodes $\{t_0, ..., t_n\}$ in $[a, b]$, we define

$$D = \det(u_k(t_l))_{k=0}^{n}\,{}_{l=0}^{n}.$$

For each $i \in \{0, ..., n\}$, another determinant $D_i(t)$ is also defined by replacing in D the entries $u_0(t_i), ..., u_n(t_i)$ in the ith column respectively by the entries $u_0(t), ..., u_n(t)$. Clearly, we have

$$y_i(t) = D^{-1} D_i(t). \tag{4}$$

We also write V for the Vandermonde matrix, in which the functions $u_0, ..., u_n$ are replaced by the monomials $1, t, ..., t^n$ and similarly define $V_i(t)$ for $i \in \{0, ..., n\}$. We note that the function $U_i(t)$, defined by

$$U_i(t) = D_i(t)[V_i(t)]^{-1}, \qquad i \in \{0, ..., n\}, \tag{5}$$

in fact depends upon the nodes $t_0, ..., t_n$, with the exception of t_i, as well as depending upon t. Furthermore, U_i is symmetric in these variables and, by the hypothesis that $\mathbf{Y}$ is spanned by an extended Tchebycheff system, is defined, continuous, and never zero on $[a, b]^{n+1}$. Writing for $j \in \{0, ..., n\}$ the expression $U_i(t; t_j)$ to emphasize the dependence of $U_i(t)$ upon t_j while t and all of the other variables are held fixed, we note that

$$U_i(t; t_j)|_{t_j = x} = U_j(t; t_i)|_{t_i = x}. \tag{6}$$

In other words, U_i in its dependence upon t_j and U_j in its dependence upon t_i are the same function.

We now rewrite (4) for $j \in \{0, ..., n\}$,

$$\begin{aligned} y_j(t) &= VD^{-1} D_j(t)[V_j(t)]^{-1} V_j(t) V^{-1} \\ &= [U_j(t_j)]^{-1} U_j(t)\, V_j(t) V^{-1}, \end{aligned} \tag{7}$$

noting that the expression $V_j(t) V^{-1}$ is simply the jth fundamental

Lagrange interpolating polynomial of degree n on the given set of nodes, with the more usual expression

$$\prod_{\substack{k=0\\k\neq j}}^{n} \frac{(t-t_k)}{(t_j-t_k)} = V_j(t)\, V^{-1}. \tag{8}$$

Returning now to consideration of the derivative matrix (1), using formula (2) to write the entries, we have

$$(-y_j(T_i)\, X_i'(t_j))_{i=1}^{n}\,{}_{j=1}^{n-1}. \tag{9}$$

Using (7), we may rewrite (9) as

$$(-[U_j(t_j)]^{-1} U_j(T_i)\, V_j(T_i)\, V^{-1} X_i'(t_j))_{i=1}^{n}\,{}_{j=i}^{n-1}. \tag{10}$$

At this point, we may, keeping (8) in consideration, multiply the jth row of the matrix for $j \in \{1, ..., n-1\}$ by

$$U_j(t_j) \prod_{\substack{k=0\\k\neq j}}^{n} (t_j-t_k)$$

and divide the ith column for $i \in \{1, ..., n\}$ by

$$(T_i-t_0)\cdots(T_i-t_n),$$

the effect of all of which is to reduce (10) to the form

$$(U_j(T_i)(t_j-T_i)^{-1} X_i'(t_j))_{i=1}^{n}\,{}_{j=1}^{n-1}. \tag{11}$$

Now, in the case that $\mathbf{Y}$ is the space spanned by polynomials of degree n or less [6], the functions $U_0, ..., U_n$ are all constant, and therefore the matrix (11) is already in the form (3), with

$$q_i(t) = (t-T_i)^{-1} X_i'(t), \qquad i \in \{1, ..., n\}.$$

In other spaces, such as those of incomplete polynomials [8], the functions $U_1, ..., U_n$ were so constituted that $U_1(t) = \cdots = U_n(t)$ for all $t \in [a, b]$, enabling their immediate cancellation from (11). In the general case under consideration here, the ith column of (11) may, for $i \in \{1, ..., n\}$, be divided by the product $U_1(T_i) \cdots U_{n-1}(T_i)$, resulting in a matrix of the form

$$(Q(T_i; t_j)(t_j-T_i)^{-1} X_i'(t_j))_{i=1}^{n}\,{}_{j=1}^{n-1}, \tag{12}$$

in which the function Q is defined by

$$Q(t; t_j) = \prod_{\substack{k=1\\k\neq j}}^{n-1} (U_k(t))^{-1} \qquad \text{for} \quad j \in \{1, ..., n-1\}.$$

By (6), $Q(t, s)$ is well-defined as a function on $[a, b]^2$. The matrix in the form (12) is at last an evaluation matrix of the form (3), with

$$q_i(t) = Q(T_i; t)(t - T_i)^{-1} X_i'(t), \qquad \text{for} \quad i \in \{1, ..., n\}, \tag{13}$$

and this concludes the proof of the theorem.

Proof of the corollary. We begin by obtaining explicit expressions for the fundamental functions. Since they are polynomials, it is possible to avoid the abstraction of Cramer's Rule and Vandermonde determinants. We may write explicitly

$$y_i(t) = \prod_{\substack{j=0\\j\neq i}}^{n} \frac{t - t_j}{t_i - t_j} \frac{f_i(t)}{f_i(t_i)}.$$

The functions $f_l(t)$ are symmetric with respect to $\{t_0, ..., t_n\} \backslash \{t_l\}$. For $j \neq l$, we write $f_l(t; t_j)$ to denote that t_j is the independent variable, the others being held constant. The important identity

$$f_i(t; t_j)|_{t_j = s} = f_j(t; t_i)|_{t_i = s} \tag{14}$$

is a restatement in our particular context of (6).

Using equivalence (2) to rewrite matrix (1), we may carry out the matrix manipulations outlined in the previous section, reducing matrix (1) in this context to the matrix

$$\left(\left(\prod_{\substack{l=1\\l\neq j}}^{n-1} f_l(T_i; t_j)\right)^{-1} \frac{X_i'(t_j)}{t_j - T_i}\right)_{i=1}^{n} {}_{j=1}^{n-1},$$

which is already an evaluation matrix of form (12), as noted in the proof of the theorem above, and as implied in the immediate context by the identity (14). One further step now converts this matrix into one in which the entries are polynomials instead of rational functions. Multiplication of the jth row, for $j \in \{1, ..., n-1\}$, by the factor

$$\prod_{k=1}^{n} \prod_{\substack{l=1\\l\neq j}}^{n-1} f_l(T_k; t_j)$$

yields a matrix, again of form (3), in which we may define the entries $q_i(t_j)$, for $i \in \{1, ..., n\}$, $j \in \{1, ..., n-1\}$, by

$$q_i(t_j) = \left(\prod_{\substack{k=1 \\ k \neq i}}^{n} \prod_{\substack{l=1 \\ l \neq j}}^{n-1} f_l(T_k; t_j) \right) \frac{X_i'(t_j)}{t_j - T_i}, \tag{16}$$

in which by (14) we may regard $q_1, ..., q_n$ as polynomials evaluated at successive points $t_1, ..., t_{n-1}$ down the columns of the matrix. Only the representation of these functions changes from row to row. This completes the proof of the corollary.

Concluding Remarks

As stated in the introduction, two problems were noted in [8], upon whose solution the establishment of general results on optimal interpolation seemed to depend. The first of these problems, reduction of the matrix (1) of partial derivatives to a matrix of evaluation, has been solved in complete generality. It is hoped that the new technique may be used to solve particular problems in which algebraic complexity has caused previous methods of solution to be totally inapplicable, or to fail.

References

1. S. Bernstein, Sûr la limitation des valeurs d'un polynome $P_n(x)$ de degré n sûr tout un segment par ses valeurs en $n+1$ points du segment, *Isv. Akad. Nauk SSSR* **7** (1931), 1025–1050.
2. C. de Boor and A. Pinkus, Proof of the conjectures of Bernstein and Erdös concerning polynomial interpolation, *J. Approx. Theory* **24** (1978), 289–303.
3. L. Brutman, On the polynomial and rational projections in the complex plane, *SIAM J. Numer. Anal.* **17** (1980), 366–372.
4. L. Brutman and A. Pinkus, On the Erdös conjecture concerning minimal norm interpolation on the unit circle, *SIAM J. Numer. Anal.* **17** (1980), 373–375.
5. P. Erdös, Some remarks on polynomials, *Bull. Amer. Math. Soc.* (*N.S.*) **53** (1947), 1169–1176.
6. T. A. Kilgore, Optimization of the Lagrange interpolation operator, *Bull. Amer. Math. Soc.* (*N.S.*) **83** (1977), 1069–1071.
7. T. A. Kilgore, A characterization of the Lagrange interpolating projection with minimal Tchebycheff norm, *J. Approx. Theory* **24** (1978), 273–288.
8. T. A. Kilgore, Optimal interpolation with incomplete polynomials, *J. Approx. Theory* **41** (1984), 279–290.
9. T. A. Kilgore, Optimal interpolation with polynomials having fixed roots, *J. Approx. Theory* **49** (1987), 378–389.
10. T. A. Kilgore, A note on optimal interpolation with rational functions, *Acta Mathematica* (*Szeged*) **52** (1988), 113–116.
11. T. A. Kilgore, A lower bound on the norm of interpolation with an extended Tchebycheff system, *J. Approx. Theory* **47** (1986), 240–245.